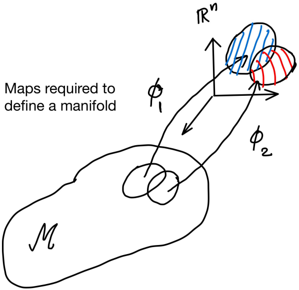
Elements of Riemannian Geometry

AGENCYLab, IUB

Aug 7, 2021

Recap

We saw that manifolds are sets whose subsets have a bijective mapping to subsets of $\mathbb{R}^n$. As there is a natural differentiable structure on $\mathbb{R}^n$ the manifold hence inherits a diffentiable structure on it too.



Now now can thus import the concept of "distance" in $\mathbb{R}^n$ to the manifold $\mathcal{M}$ via the existence of this bijection.

The metric

The metric (which is a rank-2 symmetric tensor) is given by the existence of the symmetric form :

$$ds^2 = g_{ij}(x)dx^i dx^j \Rightarrow (ds)^2 = (d\mathbf{x})^T G \ (d\mathbf{x}) \tag{1}$$

where G is a symmetric and invertible matrix representing the metric g_{ij}.

Let us consider a simple example:
In the first quadrant in the xy plane we can introduce the coordinates (u, v) :

$$xy = u; \quad y = vx \Rightarrow x = \sqrt{\frac{u}{v}}, \qquad y = \sqrt{uv} \tag{2}$$

So that $dx = \frac{1}{2}\sqrt{\frac{u}{v}}(\frac{du}{u} - \frac{dv}{v})$ and $dy = \sqrt{uv}(\frac{du}{u} + \frac{dv}{v})$

Plugging this back into $ds^2 = dx^2 + dy^2$ and carrying out the standard algebra one ends up with the matrix for the

$$G = \begin{pmatrix} \frac{\left(\sqrt{\frac{u}{v}}+\sqrt{uv}\right)}{u^2} & \frac{\left(\sqrt{uv}-\sqrt{\frac{u}{v}}\right)}{uv} \\ \frac{\left(\sqrt{uv}-\sqrt{\frac{u}{v}}\right)}{uv} & \frac{\left(\sqrt{\frac{u}{v}}+\sqrt{uv}\right)}{v^2} \end{pmatrix}$$

The presence of a metric thus allows us to define the arc length along a curve $\boldsymbol{x} = \boldsymbol{x}(\tau)$, where τ is the parameter along the arc:

$$\int_A^B d\tau \, \sqrt{g_{ij}(\boldsymbol{x}) \frac{d\boldsymbol{x}^i}{d\tau} \frac{d\boldsymbol{x}^j}{d\tau}} \tag{3}$$

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$$\int_A^B d\tau \sqrt{g_{ij}(\boldsymbol{x}) \frac{d\boldsymbol{x}^i}{d\tau} \frac{d\boldsymbol{x}^j}{d\tau}} \quad (3)$$

An interesting property to note that this expression is invariant under the reparameterization $\tau \to \tau'$, establishing that we are indeed measuring something geometrical related to the curve.

The Geodesic

Our expression (3) is valid for **any** curve. However if one is interested in the shortest curve (which is known as geodesic) between two points, one has to minimize the expression (3) under a local change

$$\boldsymbol{x} \to \boldsymbol{x'} \equiv \boldsymbol{x} + \delta \boldsymbol{x} \tag{4}$$

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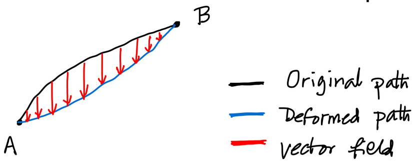

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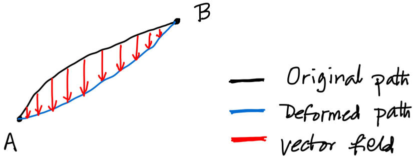

(b) The infinitesimal parameter ϵ is introduced just to emphasize that we will only keep terms first order in ϵ in our manipulations.

Let us rewrite (3) as

$$s = \int_A^B d\tau \sqrt{F}, \qquad F \equiv F[g, \boldsymbol{x}] = g_{ij}(\boldsymbol{x})\dot{x}^i \dot{x}^j \tag{5}$$

If s is a local extrema (just like 1D calculus) its change under the change (4) will be zero: $\delta s = 0$. Thus

$$\delta s = \frac{1}{2}\int d\tau\ \frac{1}{\sqrt{F}}\delta F = 0 \tag{6}$$

Now this equation shows that if we extremize instead the functional:

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$$\delta F = \frac{\partial F}{\partial g_{ij}} \delta g_{ij}(x) + 2 \frac{\partial F}{\partial \dot{x}^m} \delta \dot{x}^m = \dot{x}^i \dot{x}^j \delta g_{ij}(\boldsymbol{x}) + 2 g_{kj}(\boldsymbol{x}) \dot{x}^j \delta \dot{x}^k$$

Now some math gymnastics

Now some math gymnastics (Sorry! Tokyo Olympics is on):

$$\delta \dot{x}^m = \frac{d}{d\tau}(\delta x^m) \qquad \delta g_{ij} = \frac{\partial g_{ij}}{\partial x^k}\delta x^k$$

So that

$$\delta s' = \int_A^B d\tau \left[\dot{x}^i \dot{x}^j \frac{\partial g_{ij}}{\partial x^k}\delta x^k + 2 g_{kj}\dot{x}^j \frac{d}{d\tau}(\delta x^k)\right]$$

The second term can be integrated by parts :

$$\delta s' = \int_A^B d\tau \left[\dot{x}^i \dot{x}^j \frac{\partial g_{ij}}{\partial x^k} - 2\frac{d}{d\tau}\left(g_{kj}\dot{x}^j\right)\right]\delta x^k$$

The "boundary" term vanishes as $\delta \boldsymbol{x}$ vanish there.
So the geodesic equation can be obtained from the condition:

$$\frac{d}{d\tau}\left(g_{kj}\dot{x}^j\right) - \frac{1}{2}\dot{x}^i \dot{x}^j \frac{\partial g_{ij}}{\partial x^k} = g_{jk}\ddot{x}^k + \frac{\partial g_{jk}}{\partial x^i}\dot{x}^i \dot{x}^j - \frac{1}{2}\dot{x}^i \dot{x}^j \frac{\partial g_{ij}}{\partial x^k} = 0$$

The Scary Equation

As $\dot{x}^i \dot{x}^j$ is symmetric under the exchange of the indices $i \leftrightarrow j$ we can rewrite the 2nd term

$$\frac{\partial g_{jk}}{\partial x^i} \dot{x}^i \dot{x}^j = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} \right) \dot{x}^i \dot{x}^j .$$

Putting all of these together

$$g_{jk}(\boldsymbol{x}) \ddot{x}^k + \frac{1}{2} \left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right] \dot{x}^i \dot{x}^j = 0.$$

As $G = \{g_{jk}\}$ is an invertible matrix, we can remove the g_{jk} from the first term :

$$\ddot{x}^m + \Gamma^m_{ij} \dot{x}^i \dot{x}^j = 0 \qquad (7)$$

where

$$\Gamma^m_{ij} \equiv \frac{1}{2} g^{mk} \left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right]$$

Note g^{ij} are the matrix elements of the inverse G^{-1}.